

On the Growth of the Hurricane Depression¹

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ABSTRACT

Why do cyclones form in a conditionally unstable tropical atmosphere whose vertical thermal structure is apparently more favorable to small-scale cumulus convection than to convective circulations of tropical cyclone scale? It is proposed that the cyclone develops by a kind of secondary instability in which existing cumulus convection is augmented in regions of low-level horizontal convergence and quenched in regions of low-level divergence. The cumulus- and cyclone-scale motions are thus to be regarded as cooperating rather than as competing—the clouds supplying latent heat energy to the cyclone, and the cyclone supplying the fuel, in the form of moisture, to the clouds.

A scale-analysis indicates that it is appropriate to use the balance equations of Eliassen for the macro-motion; in this case the effect of friction in the boundary-layer may be incorporated as a condition on the vertical velocity at the top of the boundary layer. It is argued that the mean humidity in a system of convecting cumulus clouds in statistical equilibrium with the cyclone-scale circulation is appreciably less than its saturation value. The atmosphere is then gravitationally stable for the macro-scale convective process even though it is gravitationally unstable for the micro-scale convective process. The amplification of the disturbance is due to the surface frictionally induced convergence of moisture and liberation of latent heat in the center of the cyclone.

1. Introduction

Hurricanes are observed to develop from pre-existing tropical depressions that somehow have acquired a degree of circular symmetry and a warm core. However, such disturbances are rare. The typical depression is asymmetric and has a cold core. Yanai (1961) has suggested that its energy is derived from horizontal shearing motion in the easterlies and that its core is cold because of dynamically induced lifting in the presence of weak horizontal temperature gradients. This seems to be a plausible hypothesis and is in accordance with the facts. Indeed, horizontal shear instability must be produced in the equatorial convergence zone, the locus of most easterly waves, if this zone is located at some distance from the equator and the converging masses of air carry their angular momentum with them (Charney, 1963). Although the factors responsible for the transformation of a cold core system into a warm core system are not well-understood, it is known that once the transformation has taken place the depression is much more likely to amplify. What is responsible for this amplification? In the present article we shall show that surface frictional convergence will cause a small-amplitude, symmetric disturbance of depression scale in a conditionally unstable environment to amplify spontaneously. But in order to apply perturbation analysis, we

shall have to assume that the perturbation amplitude is infinitesimal, although we know that there is always, or nearly always, a pre-existing depression of small but finite amplitude. Our assumption will therefore prevent a direct application of the theoretical analysis to the actual atmosphere, and we shall have to be content with indicating the importance of processes in a small-amplitude system that we believe to be important in finite-amplitude systems as well. We wish primarily to demonstrate the importance of 1) surface friction as an energy creating mechanism and 2) the cooperative interaction of the individual cumulus cells and the large-scale motion. We have also studied finite-amplitude systems in collaboration with Dr. Y. Ogura, but here one of the basic unsolved problems is to develop a theory of surface frictional interaction at large Rossby numbers.

a) *Instability of the tropical depression. The depression and cumulus convection as cooperative phenomena.* In analogy with the theory of extratropical cyclones, it is tempting to ascribe the formation of the hurricane to some form of hydrodynamic instability, and in fact a number of such mechanisms have been proposed. The growth of disturbances has been variously attributed to the instability of temperature discontinuities (frontal instability), of horizontal temperature gradients (baroclinic instability), of radially decreasing angular momentum along isentropic surfaces in a circular vortex (rotational instability), or of horizontal shear with an extremum in the absolute vorticity profile (Rayleigh

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instability). The fourth may very well be responsible for the cold-core easterly wave but almost certainly not for the hurricane. In the opinion of the writers no good evidence has been cited to indicate that the other instabilities exist, or, if they did, that they would give rise to the observed motions. There is, however, a fifth kind of instability that undoubtedly does exist. It is the gravitational instability associated with a decrease in the entropy of saturated air with height (conditional instability). Is it possible, then, that the tropical disturbance is simply a large-scale convective overturning? This is unlikely, for, while such large-scale motions are theoretically possible, J. Bjerknes (1938) and E. Höiland (1939) have shown that conditional instability favors the smallest possible scale of cumulus convection. Indeed, were it not for friction and entrainment of dry air, the preferred convection cell would have an infinitely narrow ascending branch. And even when the motion is absolutely unstable, the theory of Bénard convection (Rayleigh, 1916), predicts roughly the same horizontal and vertical dimensions for the convection cells and hence a horizontal dimension of the order of 10 km. The energy of a cumulus cloud is derived from the unbalanced vertical pressure forces acting on buoyant elements of saturated air; there is no evidence that such forces exist on the scale of the tropical depression.

Nevertheless, conditional instability, by permitting cumulonimbus convection, must play a role in the formation of the hurricane. The most striking characteristic of the pre-hurricane depression as well as of the hurricane is the enormous rainfall in the region of low-level convergence; the latent heat energy released is two orders of magnitude greater than the amount needed to maintain the kinetic energy against frictional dissipation. This suggests that we should look upon the pre-hurricane depression and the cumulus cell not as competing for the same energy, for in this competition the cumulus cell must win; rather we should consider the two as supporting one another—the cumulus cell by supplying the heat energy for driving the depression, and the depression by producing the low-level convergence of moisture into the cumulus cell. The primary purpose of the present paper is to show that this type of interaction does lead to a large-scale self-amplification, which we may call *conditional instability of the second kind* to contrast it with the conditional instability responsible for small-scale cumulus convection.

b) *Balance in the pre-hurricane depression.* According to the present view the incipient hurricane is a *forced* circulation driven by the heat released in organized cumulus convection, not a *free* circulation driven by unbalanced buoyancy forces. We may therefore assume that the large-scale flow is quasi-hydrostatic, and we may discount unbalanced Coriolis and centrifugal forces and assume that the horizontal forces are approximately in a state of gradient or, as the case may be, geostrophic balance.

The concept of balance has been used extensively in the analysis of large-scale, gravitationally stable, extratropical motions. It has not been used for the analysis of gravitationally unstable systems because the existence of instability invalidates the assumptions under which the equations for balanced flow are derived. However, if it is realized that the instabilities give rise not to large-scale unbalanced flows but to small-scale turbulent convection which influences the large-scale flow only indirectly by transferring heat and momentum, the concept of balance may be retained. Indeed, it would seem that the use of the balance equations is made imperative by the presence of gravitational or inertial instability, for the sensitivity of the primitive Eulerian equations to the small-scale unstable modes of motion renders them unsuitable for the analysis of the large-scale motions. Thus, if one applies perturbation techniques to the study of small-amplitude perturbations of a conditionally unstable saturated atmosphere, one finds, not surprisingly, that the smallest scale modes grow at the greatest rate and ultimately predominate. Similarly, if one resorts to numerical integration, the large-scale motions are found to be obscured by the "noise" of the unstable small-scale motions.

The appropriate equations for quasi-balanced flow have been derived by Charney (1948) and Eliassen (1949) for the case where the relative horizontal velocities are small compared to the velocities of the earth's rotation (quasi-geostrophic case). More general statements of the balance equations are given in papers by Bolin (1955), Charney (1955a, b; 1962) and Thompson (1956). In the following analysis we shall assume that the flow is an axially-symmetric circular vortex. These flows were first studied by Eliassen (1952).

c) *Heating by cumulus convection.* The most difficult task in the present analysis is to describe the turbulent transport properties of the cumulus convection field in statistical equilibrium with the large-scale field of motion. Since a self-consistent theory of turbulent cumulus convection in an anisotropic mean field does not exist, one is forced to parameterize the process. This will be done by specifying a single empirical parameter, expressing the mean degree of saturation of the air in the region of active convection. If this parameter is known, and if one considers a vertical cylindrical column of air, extending through the atmosphere, whose horizontal cross-section is large enough to contain several cumulus cells and yet small enough to be regarded as infinitesimal with respect to the large-scale depression, it becomes possible to calculate the large-scale convergence of moisture into the column, and hence the rate of precipitation and latent heat supply. The latent heat will be assumed to be distributed in the vertical in proportion to the heat released from a parcel of saturated air ascending moist-adiabatically. Since the time-scale of cumulus development is small in comparison with the time-scale of development of the depression,

and the cumulus cells extend through the whole troposphere, the vertical transfer of latent heat by cumulus convection appears as an integral effect, more analogous to radiative than to convective or diffusive transfer.

d) *Surface friction as the indirect driving force.* It can be shown that time changes in a quasi-balanced flow are produced only by independent heat sources, frictional forces, or, in the case of asymmetric flow, by the transport of mass and vorticity in the balanced part of the flow. Since the motion is here assumed to be symmetric, and the condensational heating is not independent of the motion, we are led to consider the effects of surface friction. We then find that friction performs a dual role: it acts to dissipate kinetic energy, but because of the frictional convergence in the moist surface boundary layer, it also acts to supply latent heat energy to the system. The energy dissipation is proportional to the rate of working of the surface stress, whereas the energy supply is proportional to the frictional convergence, i.e., to the surface stress alone. Thus, in the early stages of the tropical depression, when the tangential velocities are small, friction acts to increase the energy of the system.

It has been shown by Charney and Eliassen (1949) that the effect of surface friction in quasi-geostrophic flow may be expressed as a boundary condition on the vertical velocity at the top of the frictional boundary layer. This device leads to a considerable simplification in the analysis. It has also been employed by Lilly (1960) in a study of convection in a conditionally unstable atmosphere, in which the disturbances are treated essentially as cumulus clouds of various sizes.

2. The balance equations for the tropical depression

It will be assumed that the pre-hurricane depression is of sufficiently small horizontal scale to permit one to ignore the curvature of the earth and the variation of the Coriolis parameter. The conditions of balance in the radial and vertical directions may then be written

$$-\rho \left(\frac{v^2}{r} + fv \right) - \frac{\partial p}{\partial r} = 0, \quad (2.1)$$

$$-\rho g - \frac{\partial p}{\partial z} = 0, \quad (2.2)$$

where p is the pressure, ρ the density, r the radial coordinate, z the vertical coordinate, v the counterclockwise tangential velocity component, f the Coriolis parameter and g the acceleration of gravity.

In the perturbation analysis (2.1) reduces to

$$\rho f v - \frac{\partial p}{\partial r} = 0, \quad (2.3)$$

i.e., the motion becomes quasi-geostrophic. For such motions Charney and Eliassen (1949) have shown that

the frictionally induced vertical velocity at the top of the Ekman layer is given by

$$w_0 = \frac{1}{2} D_E \zeta_\sigma \sin 2\alpha, \quad (2.4)$$

where $D_E = (2A/f)^{1/2}$ is a measure of the depth of the Ekman layer, A is the (constant) kinematic eddy coefficient of viscosity, α is the angle between the surface geostrophic wind and the surface isobars, and ζ_σ is the vorticity of the surface geostrophic wind. We let $A = 10 \text{ m}^2 \text{ sec}^{-1}$, a value suggested by Brunt (1939), and $f = 0.377 \times 10^{-4} \text{ sec}^{-1}$, corresponding to a latitude of 15 deg; then $D_E = 730 \text{ m}$. Since the motion is driven by surface friction, it can be shown, *a priori* by a scale analysis or *a posteriori* by a solution of the equations, that the characteristic frequency of the motion is of the order $(D_E/D)f$, so that the ratio of the neglected acceleration term in (2.1) or (2.3) to the Coriolis force is of the order $(D_E/D)^2$, where D is either the characteristic vertical scale of the motion or the scale height, and has the order of magnitude 10 km. Hence the above ratio is small ($\sim 10^{-2}$) and the balance assumption is justified.

Under the weak assumption that the characteristic phase and orbital frequencies are small compared to the frequency of sound, it may be shown that the local and horizontal fluctuations of pressure, density and temperature are fractionally small. Denoting horizontal space-time averages by bars, we may then approximate equation (2.1) by

$$\frac{\partial \bar{\chi}}{\partial r} = \frac{m^2}{r^3}, \quad (2.1')$$

where

$$\chi = (\bar{p} - \bar{p}) / \bar{p} + f^2 r^2 / 8, \quad (2.5)$$

and m is the angular momentum per unit mass:

$$m = rv + fr^2/2. \quad (2.6)$$

The continuity equation may also be approximated by

$$\frac{\partial}{\partial r} (\bar{\rho} r u) + \frac{\partial}{\partial z} (\bar{\rho} r w) = 0, \quad (2.7)$$

where u is the radial velocity.

Let the potential temperature θ be defined by

$$\ln \theta = -\frac{1}{\gamma} \ln p - \ln \rho + \text{constant}, \quad (2.8)$$

where γ is the ratio of the specific heats c_p/c_v . If the quantity $D \partial \ln \bar{\theta} / \partial z$, representing the fractional change of θ in the depth D , is small, we find that

$$\ln \bar{\theta} \simeq \ln \bar{\theta} + \frac{1}{g} \frac{\partial \chi}{\partial z}. \quad (2.9)$$

By (2.7), the divergence forms of the laws of conservation of angular momentum and energy may be written

$$\frac{\partial(\bar{\rho}rm^2)}{\partial t} + \frac{\partial}{\partial r}(\bar{\rho}rum^2) + \frac{\partial}{\partial z}(\bar{\rho}rw m^2) = 0, \quad (2.10)$$

and

$$\frac{\partial(\bar{\rho}r\theta)}{\partial t} + \frac{\partial}{\partial r}(\bar{\rho}ru\theta) + \frac{\partial}{\partial z}(\bar{\rho}rw\theta) = \frac{\bar{\rho}r\bar{\theta}Q}{c_p\bar{T}}, \quad (2.11)$$

where Q is the rate of external heating per unit mass.

We note that in (2.1), (2.3) and (2.10) no frictional forces appear. This is because we have assumed that the entire effect of friction is confined to a surface boundary layer. Although the turbulent moist-convective process which transports latent heat must also transport momentum and sensible heat both vertically and radially, these transports, unlike friction in the surface layer, are essentially dissipative and can only cut down the rate of development. As long as the vertical and radial gradients of momentum and potential temperature remain small, as they do in the free atmosphere, they will be of secondary importance. The circumstances are otherwise near the eye-wall of a hurricane, but this is another problem.

3. Heating by condensation

In the active cumulus convection zone of a tropical depression the vertical transport of moisture is $\bar{\rho}w\bar{q}$, where q is the specific humidity, and the bar denotes a horizontal average over a cross-section of the micro-column referred to in Section 1c. This quantity cannot easily be related to \bar{w} and \bar{q} , since both w and q are highly fluctuating quantities whose covariance is unknown. On the other hand, the radial flux of moisture is closely given by $\bar{\rho}u\bar{q}$, for the weak horizontal gradients of q and the small horizontal extent of the cumulus cells cause the fluctuations of u and q to be uncorrelated. For this reason we may express the horizontal convergence of moisture into a vertical unit column by

$$-\int_0^\infty \frac{1}{r} \frac{\partial}{\partial r}(r\bar{\rho}u\bar{q}) dz.$$

We must suppose that the micro-motions are in a state of statistical equilibrium with respect to the macro-motion such that \bar{q} is a known fraction $\mu(r, z, t)$ of its mean saturation value $\bar{q}_s(r, z, t)$. For lack of better knowledge we shall assume that μ is a fixed function of height alone in the convection zone. This might seem at first glance an unrealistic assumption, for it might be thought that as the depression increased in intensity its convection zone would become more and more saturated and that μ would increase with time. However, it can be seen that μ cannot approach unity because of the occurrence of precipitation simultaneously with updrafts and

downdrafts; the humidity in the downdrafts must be less than 100 per cent unless the moisture released in the updrafts remains as suspended liquid water to be evaporated later in the downdrafts; since this is certainly not the case, the mean humidity at any level above the frictional updraft layer must remain less than 100 per cent.

The total convergence of moisture into a vertical unit column now becomes

$$I = -\int_0^\infty \frac{\mu}{r} \frac{\partial}{\partial r}(\bar{\rho}r\bar{u}\bar{q}_s) dz + (\bar{\mu}\bar{\rho}\bar{w}\bar{q}_s)_0 \quad (3.1)$$

where the subscript "g" denotes a value at $z=0$, the top of the friction layer.

If we ignore the small radial variation of \bar{q}_s and integrate by parts, (3.1) becomes

$$I = -\int_0^\infty \bar{\rho}\bar{w} \frac{\partial(\mu\bar{q}_s)}{\partial z} dz. \quad (3.2)$$

Assuming that the liquid water content of the air is and remains small, the corresponding release of latent heat in the column is LI , where L is the mean latent heat of condensation. Hence, by previous assumption [Section 1c],

$$\bar{\rho}Q = -\frac{L}{q_{s0}} \frac{d\bar{q}_s}{dz} I \quad (3.3)$$

inside the cumulus convection zone. Outside of this zone $Q=0$.

Since an individual cumulus cell in the convection zone of a depression extends vertically through most of the troposphere and has a time scale which is small compared to that of the macro-motion, we must suppose that surfaces separating convecting from non-convecting regions are vertical.

We have implicitly assumed that the depression forms over the tropical oceans where there is always a source of near-saturated air in the surface boundary layer. However, we shall ignore any flux of sensible heat from the water surface.

We note also that by assumption it is not necessary to specify the manner in which moisture is transported horizontally or vertically, for the release of latent heat has been related directly to the horizontal mass convergence in a vertical column and to the known saturation humidity. This may seem odd, for there is an early stage in the formation of the tropical depression when the air is far from having reached its equilibrium humidity. The implicit assumption made here, which is needed for an application of perturbation analysis, is that this stage has already been passed. Conversely, if the system should pass over dry land the depression would presumably soon disappear. Evidence of the latter

phenomenon is found in the rapid decrease of intensity of a hurricane when it passes over extended land. Since there is no great increase in surface friction, it does not seem possible to ascribe such weakening of the depression to increased frictional dissipation over land.

4. A two-level model

While the total release of latent heat into a unit column is not badly approximated by (3.1) or (3.2), the same cannot be said for the variation of the heating with height as given by (3.3), for this variation depends on the individual motions and entrainments in cumulus clouds. A detailed treatment of the vertical structure of the depression is therefore not justified. We shall instead derive a two-level model as follows.

Let us introduce the vertical coordinate $\bar{p}(z)$ together with its sea-level value $\bar{p}_0 = \bar{p}(0)$ and denote quantities at $\bar{p} = n\bar{p}_0/4$ ($n=0, 1, 2, 3, 4$) by the subscript n . Equations (2.10) and (2.11) are then written for the levels 1 and 3 by replacing vertical derivatives by centered finite differences in the \bar{p} coordinate. Noting that $\bar{p}w$ vanishes at $\bar{p}=0$, we get the following set of equations for the time changes of m and θ :

$$\frac{\partial rm_1^2}{\partial t} + \frac{\partial}{\partial r}(ru_1m_1^2) + \frac{g}{\Delta\bar{p}}(-\bar{p}_2r w_2m_2^2) = 0, \quad (4.1)$$

$$\frac{\partial rm_3^2}{\partial t} + \frac{\partial}{\partial r}(ru_3m_3^2) + \frac{g}{\Delta\bar{p}}(\bar{p}_2r w_2m_2^2 - \bar{p}_4r w_4m_4^2) = 0,$$

$$\frac{\partial r\theta_1}{\partial t} + \frac{\partial}{\partial r}(ru_1\theta_1) + \frac{g}{\Delta\bar{p}}(-\bar{p}_2r w_2\theta_2) = \frac{r\bar{\theta}_1 Q_1}{c_p \bar{T}_1}, \quad (4.2)$$

$$\frac{\partial r\theta_3}{\partial t} + \frac{\partial}{\partial r}(ru_3\theta_3) + \frac{g}{\Delta\bar{p}}(\bar{p}_2r w_2\theta_2 - \bar{p}_4r w_4\theta_4) = \frac{r\bar{\theta}_3 Q_3}{c_p \bar{T}_3},$$

where $\Delta\bar{p} = \bar{p}_0/2$.

To obtain Q , we assume that \bar{q}_s is a linear function of \bar{p} and evaluate the integral (3.2) by the trapezoidal rule, referring to the levels 0, 2 and 4. We get

$$\begin{aligned} I &= \mu \frac{\partial \bar{q}_s}{\partial \bar{p}} \Delta\bar{p} (\bar{p}_2 w_2 + \frac{1}{2} \bar{p}_4 w_4) \\ &= \mu \frac{\partial \bar{q}_s}{\partial \bar{p}} \Delta\bar{p} \left(\frac{1}{r} \frac{\partial \psi_2}{\partial r} + \frac{1}{2r} \frac{\partial \psi_4}{\partial r} \right). \end{aligned}$$

According to (3.3)

$$Q_1 = Q_3 = \frac{gL}{\bar{q}_{s4}} \frac{\partial \bar{q}_s}{\partial \bar{p}} I \sim \frac{gL}{2\Delta\bar{p}} (\bar{q}_{s3} - \bar{q}_{s1}) \frac{1}{r} \frac{\partial}{\partial r} (\psi_2 + \frac{1}{2}\psi_4). \quad (4.3)$$

The streamfunctions ψ_2 and ψ_4 in these equations are obtained from (2.7), which allows us to write:

$$\bar{p}ru = -\frac{\partial \psi}{\partial z}, \quad \bar{p}rw = \frac{\partial \psi}{\partial r}. \quad (4.4)$$

By taking finite differences, we then get

$$\begin{aligned} ru_1 &= \frac{g}{\Delta\bar{p}} \psi_2, & ru_3 &= \frac{g}{\Delta\bar{p}} (\psi_4 - \psi_2), \\ r\bar{p}_2 w_2 &= \frac{\partial \psi_2}{\partial r}, & r\bar{p}_4 w_4 &= \frac{\partial \psi_4}{\partial r}, \end{aligned} \quad (4.5)$$

since $\psi_0 = 0$.

We next differentiate (2.1)' with respect to \bar{p} and t and introduce (2.9). This gives

$$-\frac{1}{\bar{p}_2 \theta_2} \frac{\partial^2 \theta_2}{\partial r \partial t} = -\frac{1}{r^3} \frac{\partial^2 m^2}{\partial t} \sim -\frac{1}{r^2 \Delta\bar{p}} \frac{\partial}{\partial t} (m_3^2 - m_1^2). \quad (4.6)$$

Approximating m_2^2 and θ_2 by $(m_1^2 + m_3^2)/2$ and $(\theta_1 + \theta_3)/2$, respectively, and substituting from (4.1), (4.2), (4.3) and (4.5), we get the following equation in ψ_2 :

$$\begin{aligned} &\frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} [(\theta_1 - \theta_3)\psi_2 + \theta_3\psi_4] - \frac{\theta_4}{r} \frac{\partial \psi_4}{\partial r} \right\} \\ &- \frac{2\bar{\theta}_2}{RT_2 r^4} \left\{ \psi_2 \frac{\partial}{\partial r} (m_1^2 + m_3^2) - \frac{\partial}{\partial r} (m_3^2 \psi_4) + m_4^2 \frac{\partial \psi_4}{\partial r} \right\} \\ &= \frac{\Delta\bar{p}}{g} \frac{\partial}{\partial r} \left(\frac{\bar{\theta}_1}{c_p \bar{T}_1} Q_1 + \frac{\bar{\theta}_3}{c_p \bar{T}_3} Q_3 \right) \\ &= \frac{\mu L}{2} \left(\frac{\bar{\theta}_1}{c_p \bar{T}_1} + \frac{\bar{\theta}_3}{c_p \bar{T}_3} \right) (\bar{q}_{s3} - \bar{q}_{s1}) \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (\psi_2 + \frac{1}{2}\psi_4) \right], \quad (4.7) \end{aligned}$$

where $\bar{p}_2 = \bar{p}_2 R \bar{T}_2$.

The surface boundary conditions must determine m_4 , θ_4 and ψ_4 as functions of m_1 , m_3 , θ_1 , θ_3 and the surface properties. If one assumes that the eddy viscosity is small above the constant stress layer, and that the adjustment time for the friction layer is small compared to the time scale of the macro-motion, one may ignore t and z derivatives in the momentum equation and obtain

$$\bar{p}u \frac{\partial m}{\partial r} = r \frac{\partial \tau}{\partial z}, \quad (4.8)$$

where τ is the horizontal stress. Substitution from (4.4) and integration with respect to z , then gives

$$\frac{1}{r} \frac{\partial m_4}{\partial r} \psi_4 = r \tau_u, \quad (4.9)$$

so that ψ_4 is determined if one assumes some empirical relation between surface stress and surface wind. The

above formula may also be derived by a formal expansion in powers of an eddy viscosity coefficient. The boundary mass transport thus becomes proportional to the surface stress and inversely proportional to the vertical component of absolute vorticity at the top of the constant stress layer. However, when one considers the orders of magnitude of the neglected terms, it is found that the assumptions are not valid when the centrifugal force is comparable to the Coriolis force, i.e., when the Rossby number is large.

In the present case the Rossby number is small, so that the flow is quasi-geostrophic and (2.4) is valid.

5. The perturbation equations and their solution

The tropical depression is now considered to be a small perturbation of a state of relative rest. Let primes denote perturbations and bars undisturbed quantities, and assume that all quantities are averages over the micro-fluctuations. Equation (4.7) then becomes

$$r \frac{\partial}{\partial r} \left\{ \frac{1}{r} \left[(1 - \mu\kappa) \frac{\partial \psi_2'}{\partial r} - \frac{\mu\kappa}{2} \frac{\partial \psi_4'}{\partial r} \right] \right\} - \lambda^2 (\psi_2' - \frac{1}{2} \psi_4') = 0, \tag{5.1}$$

where

$$\lambda^2 = \frac{4f^2}{RT_2} \frac{\bar{\theta}_2}{\bar{\theta}_3 - \bar{\theta}_1}, \tag{5.2}$$

$$\kappa = \frac{L \left(\frac{\theta_1}{c_p \bar{T}_1} + \frac{\theta_3}{c_p \bar{T}_3} \right) (\bar{q}_{s3} - \bar{q}_{s1})}{\bar{\theta}_1 - \bar{\theta}_3} \tag{5.3}$$

$$\simeq \left(1 - \frac{d \ln \bar{\theta}_E}{d \ln \bar{\theta}} \right)_2,$$

and $\bar{\theta}_E$ is the equivalent potential temperature, $\bar{\theta} \exp(L\bar{q}_s/c_p\bar{T})$.

The surface boundary condition (2.4) may be written

$$\frac{1}{\bar{p}_4 r} \frac{\partial \psi_4'}{\partial r} = \frac{D_e \sin 2\alpha}{2r} \frac{\partial (rv_4')}{\partial r}, \tag{5.4}$$

and the perturbation momentum equation at the top of the friction layer,

$$\frac{\partial v_4'}{\partial t} + f u_4' = 0, \tag{5.5}$$

or

$$\frac{\partial v_4'}{\partial t} + \frac{gf}{r} \left(\frac{\partial \psi}{\partial \bar{p}} \right)_4 = 0. \tag{5.6}$$

Integrating (5.4) with respect to r , differentiating with respect to t , substituting from (5.6), and expressing the \bar{p} -derivative as a non-centered difference between levels

4 and 2, we get

$$\frac{\partial \psi_4'}{\partial t} = K (\psi_2' - \psi_4'), \tag{5.7}$$

where

$$K = (D_E/H_2) f \sin 2\alpha \tag{5.8}$$

and H is the scale height RT_2/g ,

We now seek solutions to (5.1) and (5.7) of the form $\psi' = \Psi(r)e^{\sigma t}$. Substitution of this expression and elimination of Ψ_4 gives

$$r \frac{d}{dr} \left(\frac{1}{r} \frac{d\Psi_2}{dr} \right) \pm \lambda_{\pm}^2 \Psi_2 = 0, \tag{5.9}$$

where the subscripts “+” and “-” refer to the regions of rising and sinking motion, respectively, and

$$\lambda_-^2 = \frac{\sigma/K + \frac{1}{2}}{\sigma/K + 1} \lambda^2, \tag{5.10}$$

$$\lambda_+^2 = \left[-1 + \kappa\mu \left(\frac{\sigma/K + \frac{3}{2}}{\sigma/K + 1} \right) \right]^{-1} \lambda_-^2.$$

In the region of rising motion the solution of (5.9) satisfying the condition $\Psi_2 = 0$ at $r = 0$ is

$$\Psi_+(r) = A_+ r J_1(\lambda_+ r), \tag{5.11}$$

where J_1 is the Bessel function of the first kind and order 1. In the region of descending motion the solution of (5.9) satisfying the condition that $ru \rightarrow 0$ as $r \rightarrow \infty$ is

$$\Psi_-(r) = A_- r H_1^{(1)}(i\lambda_- r), \tag{5.12}$$

where $H_1^{(1)}$ is the Hankel function of order 1.

At $r = a$, the radius separating rising from sinking motion, continuity of ψ implies that

$$\Psi_+(a) - \Psi_-(a) = 0, \tag{5.13}$$

and integration of (5.9) across the discontinuity gives

$$\frac{1}{\lambda_+^2} \frac{d\Psi_+(a)}{dr} + \frac{1}{\lambda_-^2} \frac{d\Psi_-(a)}{dr} = 0. \tag{5.14}$$

This condition may also be derived from continuity in pressure. Together, (5.13) and (5.14) yield the eigenvalue equation

$$\frac{J_1(\lambda_+ a)}{J_0(\lambda_+ a)} = i \frac{\lambda_- H_1^{(1)}(i\lambda_- a)}{\lambda_+ H_0^{(1)}(i\lambda_- a)}, \tag{5.15}$$

relating σ/K to the parameters a and $\kappa\mu$.

There is some arbitrariness in the choice of κ since $\bar{\theta}$ and \bar{q}_s are not uniformly variable and yet must be assumed so in the approximation. We postulate uniform conditions in the entire interval $\bar{p}_4 \leq \bar{p} \leq 0$ and choose $(\bar{\theta}_3 - \bar{\theta}_1)/\bar{\theta}_2$ and $\bar{q}_{s3} - \bar{q}_{s1}$ as characteristic of the con-

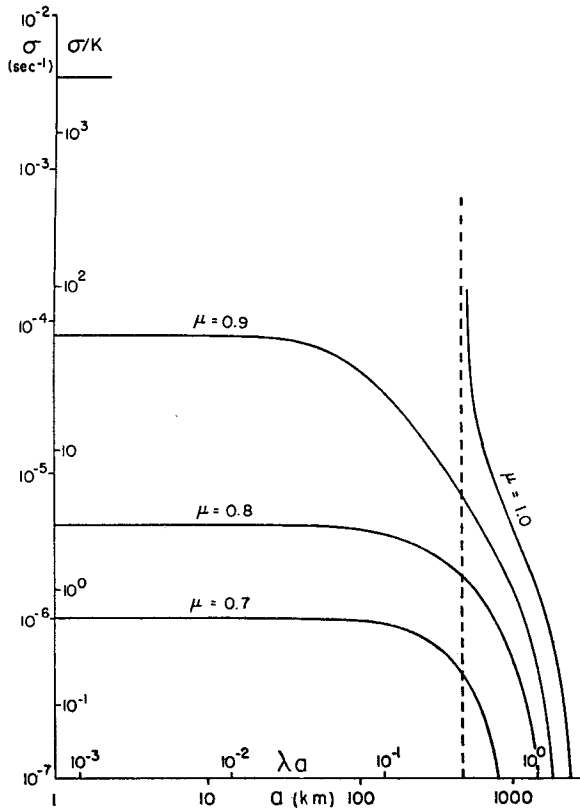


FIG. 1. The growth rate of the tropical depression as a function of the radius of the cloud region.

ditionally unstable part of the troposphere above the friction layer. From the conditions of the tropical atmosphere in the hurricane season at latitude 15 deg we find $\kappa \approx 1.1$.

6. Discussion

Fig. 1 contains graphs computed from equation (5.15) of σ/K as a function of $a\lambda$ for different values of μ . Taking $f = 0.377 \times 10^{-4} \text{ sec}^{-1}$, $\alpha = 15^\circ$, $\bar{p}_0 = 1000 \text{ mb}$, $H = 8.0 \text{ km}$, we find $D_E = \sqrt{2A/f} = 0.73 \text{ km}$ and $K = \sin 2\alpha (D_E/H) f = 1.72 \times 10^{-6} \text{ sec}^{-1}$. These values are used to determine the scales of a and σ .

It is seen from the figure that the maximum value of σ lies in the range 10^{-6} – 10^{-5} sec^{-1} , giving an e -folding time in the range 10 days to 1 day for μ in the range 0.7 to 0.8. The growth rates become much higher as μ approaches 1, and in fact become infinite for finite a when $\mu > 1/\kappa = 0.91$. However, values of μ larger than 0.8 appear to be unrealistic. If it is assumed that μ is 0.8, the e -folding time, $\sigma^{-1} = 2.5$ days and the radius, $a \approx 100 \text{ km}$, at which the growth rate levels off seem to be reasonable values for the tropical depression. At very much smaller values of a , friction no longer acts to destabilize the system, and σ should in reality decrease, as it would if K were to tend toward zero with

a . We may therefore accept the order of magnitude $a = 100 \text{ km}$ as the size of the active convective region of the unstable disturbance.

The growth rate $\sigma = 4.6 \times 10^{-6} \text{ sec}^{-1}$ corresponding to $\mu = 0.8$ is in marked contrast to the value $(-g\theta \ln \bar{\theta}_E / \partial z)^{1/2} \approx 4 \times 10^{-3} \text{ sec}^{-1}$ (indicated in the upper left corner of the figure by a horizontal bar), which is the growth rate approached by the cumulus cell as the updraft radius diminishes to zero. We are dealing with two physically distinct phenomena, with simple gravitational instability in a moist environment on the one hand, and with the frictionally driven, cooperative, depression-cumulus system on the other.

Acknowledgment. This article is a continuation of work presented by Charney at the Technical Conference on Hurricanes of the American Meteorological Society in Miami Beach, Fla., 19–22 November 1958, and elaborated at the 40th Anniversary Meeting of the American Meteorology Society in Boston, 19–22 January 1960. In this work it was suggested that the tropical depression and the associated cumulus cloud system can be treated as cooperative phenomena with the aid of the balance equations, and that surface friction acts as an energy producing mechanism. However, there remained an essential defect in the theory: it was assumed that the mean relative humidity in the region of low-level convergence was 100 per cent. Such a condition can be attained if there are no downdrafts surrounding the rising parcels of saturated air and no entrainment, but this is a physically unrealistic assumption. As a result of conversations with Dr. K. Ooyama, the authors were led to consider more carefully the properties of the cumulus cells in statistical equilibrium with the macro-motion, and thereby to the conclusion that the mean atmosphere is unsaturated in the region of mean rising motion and therefore can remain gravitationally stable for the macro-motion even though it is gravitationally unstable for the micro-(cumulus) motions.

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